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Valores Propios de la Matriz de Truncamiento Asociados al Operador de Transición de la Máquina Sumadora en la base 2.

Eigenvalues of Truncation Matrix Associated to the Transition Operator of the Adding Machine in Base 2.

Ali Messaoudi * and Rafael Asmat Uceda †

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Resumen

En este trabajo construimos la máquina sumadora estocástica en la base 2 considerando la matriz de truncamiento S_n asociada al operador S y estudiamos los valores propios de la matriz S_n actuando en $l^\infty(\mathbb{N})$.

Palabras claves. Estocástico, máquina sumadora, valores propios

Abstract

In this work we build the stochastic adding machine in base 2 considering the truncation matrix S_n associated to the operator S and we study the eigenvalues of the matrix S_n acting in $l^\infty(\mathbb{N})$.

Keywords. Stochastic, adding machine, eigenvalues

1. Introduction. Given a natural number N we can write it, in a unique way, by using the greedy algorithm as $N = \sum_{i=0}^{k(N)} \varepsilon_i(N)2^i$, where $(\varepsilon_i)_{i\geq 0}$ are the digits of N in base 2 taking the values 0 or 1. There exists an algorithm (adding machine) that computes the digits of $N+1$. We summarize this process in terms of a system of evolving equation, introducing an auxiliary variable "carry 1", $c_i(N)$, for each digit $\varepsilon_i(N)$, in the following way:

Put $c_{-1}(N+1) = 1$ and

$$\varepsilon_i(N+1) = (\varepsilon_i(N) + c_{i-1}(N+1))mod(2)$$

$$c_i(N+1) = \left[\frac{\varepsilon_i(N) + c_{i-1}(N+1)}{2}\right]$$

where $i \geq 0$ and $[z]$ is the integer part of $z \in \mathbb{R}^+$.

P. R. Killeen and T. J. Taylor [9] built a stochastic adding machine considering an independent, identically distributed family of random variables $\{e_i(n) : i \geq 0, n \in \mathbb{N}\}$ parameterized by the natural numbers i and n. This family take the value 0 with probability $1-p$ and the value 1 with probability p. Given a natural number N we consider the sequences $(\varepsilon_i(N+1))_{i\geq 0}$ and $(c_i^{'}(N+1))_{i\geq -1}$ defined por $c_{-1}^{'}(N+1) = 1$ and for all $i \geq 0$

*Departamento de Matemáticas, Universidade Estadual Paulista, Julio de Mesquita Filho, São José do Rio Preto, São Paulo - Brasil. *messaoud@ibilce.unesp.br*.

†Departamento de Matemáticas, Universidad Nacional de Trujillo, Av. Juan Pablo II s/n., Ciudad Universitaria, Trujillo-Perú. *rasmat@unitru.edu.pe*.

$$\varepsilon_i(N+1) = (\varepsilon_i(N) + e_i(N)c^{'}_{i-1}(N+1))mod(2)$$

$$c^{'}_i(N+1) = \left[\frac{\varepsilon_i(N) + e_i(N)c^{'}_{i-1}(N+1)}{2}\right].$$

Hence, a number $\sum_{i=0}^{+\infty} \varepsilon_i(N)2^i$ transitions to a number $\sum_{i=0}^{+\infty} \varepsilon_i(N+1)2^i$. In a equivalent way, we obtain a Markov process $\psi(N)$ with state space $\mathbb{N}$ by doing $\psi(N) = \sum_{i=0}^{+\infty} \varepsilon_i(N)2^i$.

Killeen and Taylor [9] study the spectrum of the transition operator S acting in $l^{\infty}(\mathbb{N})$. They prove that the spectrum $\sigma(S)$ of S is connected to the Julia set of the quadratic map $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = (z - (1-p))^2/p^2$.

In particular, an integer with binary representation $\varepsilon_n \dots \varepsilon_k 0 \underbrace{11\dots 11}_{k}$ transitions to the integer $\varepsilon_n \dots \varepsilon_{k+1} 1 \underbrace{00\dots 00}_{k}$ with probability p^{k+1} and an integer with binary representation $\varepsilon_n \dots \varepsilon_k \underbrace{11\dots 11}_{k}$ transitions to $\varepsilon_n \dots \varepsilon_k 0 \underbrace{00\dots 00}_{k}$ with probability $p^k(1-p)$.

In this paper, we consider the truncation matrix S_n associated to the operator S, where $(S_n)_{i,j} = S_{ij}$ for all $0 \le i, j \le n$ and we study the set of eigenvalues, $\sigma(S_n)$. In particular, we prove:

Theorem 3: Let $E = (\bigcup_{k=1}^{\infty} \sigma(S_{2^k}))'$, where A' denotes the set of accumulation points of A. Then, the set E satisfies:

1) If $0 < p < 1$, then $\partial J_c(f) \subset E$.
2) If $0 < p < 1/2$, then $E = \partial J_c(f)$.
3) If $1/2 \le p < 1$, then $E \subset J_c(f)$.

Theorem 4: The spectrum of the transition operator S acting in $l^{\infty}(\mathbb{Z})$ is equal to the point spectrum of S and it is equal to the Julia set of the quadratic function $f(z) = \left(\frac{z-(1-p)}{p}\right)^2$.

The first section is devoted to define the stochastic adding machine in base 2 by using the idea of transductors and in the next section we prove some results by considering the truncation matrix, S_n of the operator S associated to the stochastic adding machine.

2. Transductor and Stochastic Adding Machine. It is know that the addition of 1 in base 2 is given by a finite transductor on $A^* \times A^*$, where $A = \{0, 1\}$ is a finite alphabet and A^* is the set of finite words on A. The idea of the adding machine by using tranductors was introduced by Messaoudi-Smania [11]. This tranductor is composed by two states, an initial state, I, and a terminal one, T. The initial state is connected to itself by one arrow labeled by $(1/0)$. There is also one arrow going from the initial state to the terminal one. This arrow is labeled by $(0/1)$. The terminal state is connected to itself by one arrow labeled by (x/x).

Let assume $N = \varepsilon_n \dots \varepsilon_0$. To find the digits of $N+1$ let consider the finite path $c = (p_{k+1}, a_k/b_k, p_k) \dots (p_2, a_1/b_1, p_1)(p_1,$ where $p_i \in \{I, T\}, p_0 = I, p_{k+1} = T, a_i, b_i \in A^*$.

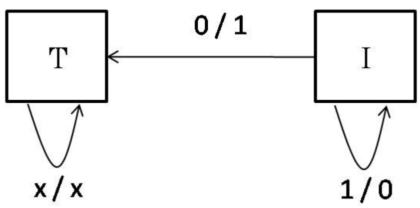


Figure 1: Transductor of the adding machine in base 2

Furthermore $\dots 0 \dots 0 a_k \dots a_0 = \dots 0 \dots 0 \varepsilon_n \dots \varepsilon_0$.

Hence $N + 1 = \varepsilon^{'}_n \dots \varepsilon^{'}_0$, where

$$\dots 0 \dots b_k \dots b_0 = \dots 0 \dots 0 \varepsilon^{'}_n \dots \varepsilon^{'}_0$$

Example 1. *If $N = 11 = 1011_2$ then N corresponds to the path*

$$(T, 1/1, T)(T, 0/1, I)(I, 1/0, I)(I, 1/0, I).$$

Hence $N + 1 = 1100_2 = 12$.

If $N = 13 = 1101_2$ then N corresponds to the path

$$(T, 1/1, T)(T, 1/1, T)(T, 0/1, I)(I, 1/0, I).$$

Then, $N + 1 = 1110_2 = 14$.

2.1. Stochastic Adding Machine in Base 2. It is possible to generalize this description to construct a probabilistic transductor $\mathcal{T}_p,\ 0 < p < 1$, of the stochastic adding machine in base 2 (see [11]) as showed in the figure 2.1.

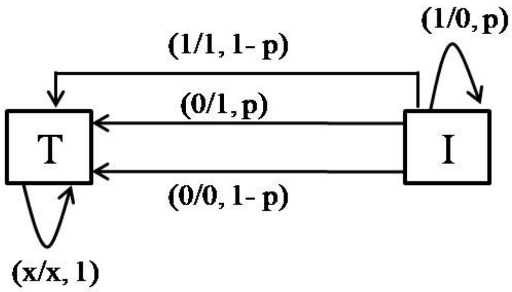


FIGURA 2.1. *Transductor of the stochastic adding machine in base 2*

Example 2. *If $N = 11 = 1011_2$, then N in the transductor of adding machine, N corresponds to the path,*

$$(T, 1/1, T)(T, 0/1, I)(I, 1/0, I)(I, 1/0, I).$$

In the stochastic adding machine we have the following paths:

(a) $(T,(1/1,1),T)(T,(0/0,1),T)(T,(1/1,1),T)(T,(1/1,1-p),I)$. *In this case $N = 1011$ transitions to 1011 with probability $1-p$.*

(b) $(T,(1/1,1),T)(T,(0/0,1),T)(T,(1/1,1-p),I)(I,(1/0,p),I)$. *In this case $N = 1011$ transitions to 1010 = 10 with probability $p(1-p)$.*

(c) $(T,(1/1,1),T)(T,(0/0,1-p),I)(I,(1/0,p),I)(I,(1/0,p),I)$. *In this case $N = 1011$ transitions to 1000 = 8 with probability $p^2(1-p)$.*

(d) $(T,(1/1,1),T)(T,(0/1,p),I)(I,(1/0,p),I)(I,(1/0,p),I)$. *In this case $N = 1011$ transitions to 1100 = 12 with probability p^3.*

By using the transductor $\mathcal{T}_p$ we have the following result.

Proposition 1. *Let N be a non-negative integer, then the following results are satisfied.*

i. *N transitions to N with probability $1-p$.*

ii. *If $N = \varepsilon_k \dots \varepsilon_1 0, k \geq 1$ (even case), then N transitions to $N+1$ with probability p.*

iii. *If $N = \varepsilon_k \dots \varepsilon_t 0 \underbrace{1 \dots 1}_{s}, s \geq 1, k \geq t \geq s+1$ (odd case), then N transitions to $N + 1 = \varepsilon_k \dots \varepsilon_t 1 \underbrace{0 \dots 0}_{s}$ with probability p^{s+1} and N transitions to $N - 2^m + 1 = \varepsilon_k \dots \varepsilon_k 0 \underbrace{1 \dots 1}_{s-m} \underbrace{0 \dots 0}_{m}$, $1 \leq m \leq s$ with probability $p^m(1-p)$.*

Proof:

(i) If $N = \varepsilon_k \dots \varepsilon_1 0$ then N corresponds to the path (in the adding machine in base 2)

$$(T, \varepsilon_k/\varepsilon_k, T) \dots (T, \varepsilon_1/\varepsilon_1, T)(T, 0/1, I).$$

In the stochastic adding machine in base 2 N corresponds to the following paths:

(A) $(T,(\varepsilon_k/\varepsilon_k,1),T) \dots (T,(\varepsilon_1/\varepsilon_1,1),T)(T,(0/0,1-p),I)$. In this case N transitions to N with probability $1-p$.

(B) $(T,(\varepsilon_k/\varepsilon_k,1),T)\ldots(T,(\varepsilon_2/\varepsilon_2,1),T)(T,(0/1,p),I)$. Hence N transitions to $N+1$ with probability p.

(ii) If $N=\varepsilon_k\ldots\varepsilon_t 0\underbrace{1\ldots 1}_{s}$, then N corresponds to the path

$(T,\varepsilon_k/\varepsilon_k,T)\ldots(T,\varepsilon_t/\varepsilon_t,T)(T,0/1,I)\underbrace{(I,1/0,I)\ldots(I,1/0,I)}_{s}$. In the stochastic adding machine in base 2, N corresponds to the paths.

(A) $(T,(\varepsilon_k/\varepsilon_k,1),T)\ldots(T,(\varepsilon_t/\varepsilon_t,1),T)(T,(0/0,1),T)$ $\underbrace{(T,1/1,1),T)\ldots(T,(1/1,1-p),I)}_{s}$. In this case N transitions to N with probability $1-p$.

(B) $(T,(\varepsilon_k/\varepsilon_k,1),T)\ldots(T,(\varepsilon_t/\varepsilon_t,1),T)(T,(1/0,p),I)$ $\underbrace{(I,(0/1,p),I)\ldots(I,(0/1,p),I)}_{s}$. In this case N transitions to $N+1$ with probability p^{s+1}.

(C) $(T,(\varepsilon_k/\varepsilon_k,1),T)\ldots(T,(\varepsilon_t/\varepsilon_t,1),T)(T,(0/0,1),T)$ $\underbrace{(T,(1/1,1),T)\ldots(T,(1/1,1),T)}_{s-m-1}(T,(1/1,1-p),I)$ $\underbrace{(I,(1/0,p),I)\ldots(I,(1/0,p),I)}_{m}$. In this case N transitions to

$$N-2^m+1=\varepsilon_k\ldots\varepsilon_t 0\underbrace{1\ldots 1}_{s-m}\underbrace{0\ldots 0}_{m}$$

with probability $p^m(1-p)$.

□

By using the proposition 1, we build the transition graph,

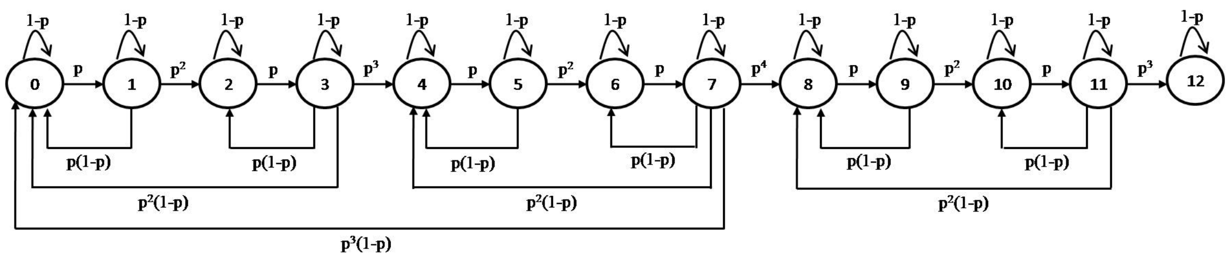


FIGURA 2.2. *Transition graph of the stochastic adding machine in base 2*

Observation 1. *We can note that the transition graph is self-similar, since it is made up of blocks repeating themselves in a periodic way.*

The transition operator S_p, associated to the transition graph is given by:

$$S_p = \begin{pmatrix}
1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \dots \\
p(1-p) & 1-p & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \dots \\
0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \dots \\
p^2(1-p) & 0 & p(1-p) & 1-p & p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \dots \\
0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \dots \\
0 & 0 & 0 & 0 & p(1-p) & 1-p & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 \dots \\
p^3(1-p) & 0 & 0 & 0 & p^2(1-p) & 0 & p(1-p) & 1-p & p^4 & 0 & 0 & 0 & 0 & 0 \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p(1-p) & 1-p & p^2 & 0 & 0 & 0 \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2(1-p) & 0 & p(1-p) & 1-p & p^2 & 0 \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p(1-p) & 1-p \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & &
\end{pmatrix}$$

Observation 2.

1. *We can verify that the sum of coefficients of every line (resp. every column) of S_p is equal to 1. Then, the transition matrix, S_p, is a doubly stochastic matrix.*
2. *We can note that exists a matrix $A = \begin{pmatrix} 1-p & p \\ p(1-p) & 1-p \end{pmatrix}$ that repeats itself infinitely in the diagonal region.*
3. *Since S_p is a stochastic matrix, then $S_p(l^\infty) \subset l^\infty$. In effect, if $v = (v_i) \in l^\infty$, then $|S_p v|_\infty = \sup_{i\in\mathbb{N}} |S_p v_i| = \sup_{i\in\mathbb{N}} \left|\sum_{j=0}^{\infty} S_{ij} v_j\right| \leq |v|_\infty$.*
4. *If $p = 1$, then the matrix S_p is defined by*

$$\begin{cases} S_{ij} = 1, & j = i+1 \\ S_{ij} = 0, & j \neq i+1 \end{cases}$$

Hence we have $S \begin{pmatrix} x_0 \\ x_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}$. In this case, the matrix S_p corresponds to the shift operator. If $p \neq 1$ we can consider the operator S_p as a stochastic perturbation of the shift operator.

Theorem 1. *(Abdalaoui-Messaoudi [1]) The spectrum of S in $l^\infty(\mathbb{N})$ is equal to the filled Julia set of the quadratic function $f(z) = \left(\frac{z-(1-p)}{p}\right)^2$, that is,*

$$\sigma_p = \{z \in \mathbb{C}, (f^n(z))_{n\geq 0} \text{ is bounded}\}$$

where $f^n = \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}$, $\forall\, n \in \mathbb{N}$.

Observation 3. *Consider the transition operator $\tilde{S} = (S_p - (1-p)I)/p$. Then $\tilde{S}$ has a period 2 and the operator $\tilde{S}^2$ is decomposed in two transition graphs, one of them acting on even natural number and the other one acting on odd natural number and every graph is isomorphic to the transition graph of S_p.*

Theorem 2. *(Abdalaoui-Messaoudi [1]) The operator S_p is defined in the classical Banach space, $C_0(\mathbb{N})$ and $l^q(\mathbb{N}), q \geq 1$. In particular particular, the spectrum of the operator S_p acting in $\mathcal{C}_0(\mathbb{N})$ (resp. $l^q(\mathbb{N}), q > 1$) is equal to the continuum spectrum of S_p and equal to the filled Julia set of f, $J_c(f)$. On $l^1(\mathbb{N})$, the point spectrum of S_p is empty, the residual spectrum contains a dense and countable subset of the Julia set, $\partial J(f)$. The continuum spectrum is equal to $J_c(f)\backslash\sigma_r(S)$.*

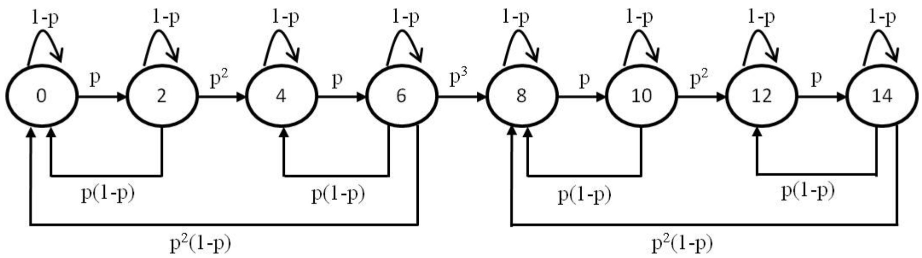


FIGURA 2.3. *Transition graph of the stochastic adding machine on even natural numbers*

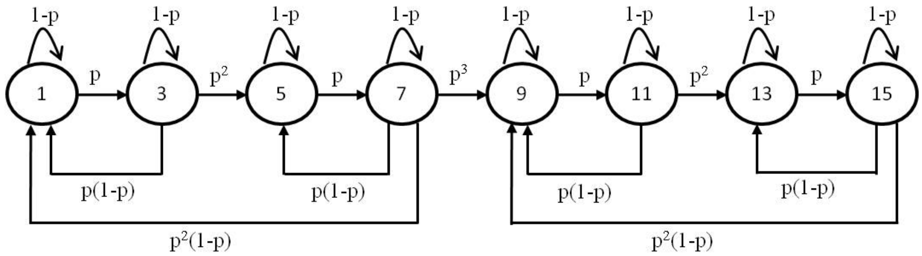


FIGURA 2.4. *Transition graph of the stochastic adding machine on odd natural numbers*

3. Results. Let consider a natural integer number n and denote S_n the matrix $n \times n$, where

$$(S_n)_{i,j} = S_{ij},\ \forall\, 0 \leq i, j \leq n$$

S_n is the truncation matrix of S. $\sigma(S_n)$ denotes the set of the eigenvalues of S_n.

Definition 1. *Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ be. We define the sequence of complex numbers $(q_n(\lambda))_{n\geq 0}$ given by:*

$$q_1 = -\frac{1-p-\lambda}{p},\quad q_{2^n} = \frac{1}{p}q^2_{2^{n-1}} - \left(\frac{1}{p} - 1\right),\ \forall\, n \geq 1$$

Theorem 3. *Let $E = (\bigcup_{k=1}^{\infty}\sigma(S_{2^k}))'$. Then E satisfies the following:*
i. If $0 < p < 1$, then $\partial J_c(f) \subset E$.
ii. If $0 < p < 1/2$, then $E = \partial J_c(f)$.
iii. If $1/2 \leq p < 1$, then $E \subset J_c(f)$.

To prove Theorem 3 we need some results.

Proposition 2. *Let $k \in \mathbb{N}$, then $\sigma(S_{2^k}) = \{\lambda \in \mathbb{C}, q_{2^k}(\lambda) = 0\}$, where $(q_n)_{n\geq 1}$ is the sequence defined above.*

Proof: Let $\lambda \in \sigma(S_{2^k})$, then there exists $v = (v_i)_{0\leq i \leq 2^k-1} \in \mathbb{C}^{2^k} \setminus \{0\}$ such that $(S_{2^k} - \lambda I)v = 0$. It is possible to prove that, $v_i = q_i v_0$, for all $1 \leq i \leq 2^k - 1$ and furthermore,

$$\sum_{i=0}^{2^k-2} p_{2^k-1,i}v_i + (1-p-\lambda)v_{2^k-1} = 0. \tag{1}$$

On the other hand, let consider the system

$$\begin{cases} (1-p-\lambda)w_0 + pw_1 = 0 \\ p(1-p)w_0 + (1-p-\lambda)w_1 + p^2 w_2 = 0 \\ \vdots \\ \displaystyle\sum_{i=0}^{2^k-2} p_{2^k-1,i}w_i + (1-p-\lambda)w_{2^k-1} + p_{2^k-1,2^k}w_{2^k} = 0 \end{cases}$$

where $w_0 = v_0$.

Thus, we have, $w_i = v_i = q_i v_0, \forall\, 1 \leq i < 2^{k-1}$, e $w_{2^k} = q_{2^k}w_0 = q_{2^k}v_0$.

Hence,

$$0=\sum_{i=0}^{2^k-2} p_{2^k-1,i}v_i-(1-p-\lambda)v_{2^k-1}=\sum_{i=0}^{2^k-2} p_{2^k-1,i}w_i-(1-p-\lambda)w_{2^k-1}$$

$$=-p_{2^k-1,2^k}w_{2^k}=-p_{2^k-1,2^k}q_{2^k}v_0=p^{k+1}q_{2^k}v_0.$$

Hece, $q_{2^k}=0$. Therefore $\sigma(S_{2^k})\subset\{\lambda\in\mathbb{C},q_{2^k}(\lambda)=0\}$.

On the other hand, from $q_{2^k}(\lambda)=0$ we have (1) is true. Hence we are done. □

Lemma 1.

$$E=\{z\in\mathbb{C},\exists\,(w_{n_i})_{i\geq 0}\subset\mathbb{C},w_{n_i}\ \textit{2 - 2 differents such that}\ \lim_{i\to\infty} w_{n_i}=z\ \textit{and}\ f^{n_i}(w_{n_i})=0\}.$$

Proof: Let $\lambda\in\mathbb{C}$, then $f(\lambda)=q_1^2(\lambda)$. Hence

$$f^{n+1}(\lambda)=q_{2^n}^2(\lambda),\ \forall n\in\mathbb{N} \tag{2}$$

Let $z\in E$, then there is a sequence $(w_{n_i})_{i\geq 0}\subset\bigcup_{k=1}^{+\infty}\sigma(S_{2^k})$ such that $\lim_{i\to\infty} w_{n_i}=z$. For every $i\in\mathbb{N}$ there exists $n_i-1\in\mathbb{N}$ such that

$$q_{2^{n_i-1}}(w_{n_i})=0.$$

Hence by (2) we have $f^{n_i}(w_{n_i})=0$.

The other inclusion is clear. □

Proposition 3. *The following results are true:*

i. $1\in\partial J_c(f)$.
ii. $1\in E$.
iii. $f(E)\subset E$.
iv. $f^{-1}(E)\subset E$.

Proof:

i. We have $f(1)=\left(\frac{1-(1-p)}{p}\right)^2=1$ and $|f'(1)|=|\frac{2}{p}(1-(1-p))|=2>1$. Then 1 is a repulsive fixed point of f and thus $1\in\partial J_c(f)$.

ii. Let consider the sequence w_n defined by

$$w_1=1-p,w_2=1-p+p\sqrt{1-p},w_n=1-p+p\sqrt{w_{n-1}},\ \forall n>2.$$

Claim: $(w_n)_{n\geq 0}$ is convergent and $\lim_{n\to\infty} w_n=1$. Moreover, $f^n(w_n)=0,\ \forall\, n\in\mathbb{N}^*$.

In fact: $(w_n)_{n\geq 0}$ is a increasing sequence and $w_n\leq 1,\ \forall n\in\mathbb{N}^*$. Let $L\in\mathbb{R}$ such that $\lim w_n=L$. We have $L=1-p+p\sqrt{L}$, hence $f(L)=L$. The fixed points of f are 1 and $(1-p)^2$. Since $(1-p)^2<1-p\leq w_1$, then , $L=1$.

On the other hand, we have $f(w_1)=\left(\frac{1-p-(1-p)}{p}\right)^2=0$. Assume that $f^{n-1}(w_{n-1})=0$. We have, $f(w_n)=w_{n-1}$, for all $n\in\mathbb{N}^*$ and by Lemma 1 we deduce $1\in E$.

iii. Let $z\in E$, then there is a sequence $(w_{n_i})_{n\geq 0}$ converging to z and such that $f^{n_i}(w_{n_i})=0$. Since f is continuous, $\lim_{i\to+\infty} f(w_{n_i})=f(z)$.

Let $w'_{n_i-1}=f(w_{n_i})$, then w'_{n_i-1} converges to $f(z)$ and furthermore, $f^{n_i-1}(w'_{n_i-1})=f^{n_i}(w_{n_i})=0$. Thus, $f(z)\in E$.

iv. Let $z\in f^{-1}(E)$, then $f(z)\in E$. Hence there exists a sequence w_{n_i} converging to $f(z)$ and such that $f^{n_i}(w_{n_i})=0$. Since

$$f(z)=\left(\frac{z-(1-p)}{p}\right)^2,$$

then there are $x_{n_i}, y_{n_i} \in \mathbb{C}$ such that $f^{-1}(w_{n_i}) = \{x_{n_i}, y_{n_i}\}$ and therefore

$$f(x_{n_i}) = w_{n_i} \longrightarrow f(z),\ f(y_{n_i}) = w_{n_i} \longrightarrow f(z).$$

We have,

$$f(x_{n_i}) - f(z) = \left(\frac{x_{n_i} - (1-p)}{p}\right)^2 - \left(\frac{z-(1-p)}{p}\right)^2 =$$

$$= \left(\frac{x_{n_i} - z}{p}\right)\left(\frac{x_{n_i} + z - 2(1-p)}{p}\right)$$

converges to zero whenever i goes to $+\infty$.

On the other hand,

$$x_{n_i} = \pm p\sqrt{w_{n_i}} + (1-p),$$

$$y_{n_i} = \mp p\sqrt{w_{n_i}} + (1-p).$$

Hence

$$x_{n_i} - y_{n_i} = \pm 2p\sqrt{w_{n_i}} \to \pm 2p\sqrt{f(z)} \tag{3}$$

Assume that for every subsequence $(n_i^{'})$ of (n_i) we have neither $(x_{n_i^{'}})$ or $(y_{n_i^{'}})$ converge to z. Then

$$\lim x_{n_i^{'}} = -z + 2(1-p) \text{ and } \lim y_{n_i^{'}} = -z + 2(1-p) \tag{4}$$

Hence,

$$\lim x_{n_i^{'}} - y_{n_i^{'}} = 0 \tag{5}$$

Case 1. If $f(z) \neq 0$, then by (3) and (5) we deduce that there exists a subsequence $(n_i)_{i\geq 0}$ such that $\lim x_{n_i} = z$ or $\lim y_{n_i} = z$.

Assume that $\lim\limits_{i\to+\infty} x_{n_i} = z$, then $f^{n_i+1}(x_{n_i}) = f^{n_i}(w_{n_i}) = 0$. Hence $z \in E$ and therefore $f^{-1}(E) \subset E$.

Case 2. If $f(z) = 0$, then $z = 1-p$. (6)
Since $f(z) \in E$, then there exists a subsequence w_{n_i} converging to 0 and such that $f^{n_i}(w_{n_i}) = 0$. Moreover, $f^{-1}(w_{n_i}) = \{x_{n_i}, y_{n_i}\}$ and $f(x_{n_i}) = w_{n_i} \to 0$.

From (4) and (6), we obtain $\lim x_{n_i} = z$ and $\lim y_{n_i} = z$.

Putting $x_{n_i} = x^{'}_{n_i+1}$, we have $\lim x^{'}_{n_i+1} = z$ such that $f^{n_i+1}(x^{'}_{n_i+1}) = f^{n_i}(w_{n_i}) = 0$ and therefore $z \in E$.

□

Lemma 2. $0 \in J_c(f)$ *if and only if* $p \geq 1/2$.
Proof: Assume $1/2 \leq p < 1$, then

$$0 < f(0) = \left(\frac{1}{p} - 1\right)^2 < 1,$$

$$0 \leq f^2(0) = \left(\frac{f(0) - (1-p)}{p}\right)^2 < 1,$$

Carrying on with this procedure, we deduce $0 \leq f^n(0) < 1$, for all $n \in \mathbb{N}$. Hence the sequence $(f^n(0))_{n\geq 1}$ is bounded and therefore $0 \in J_c(f)$.

Let suppose $p < 1/2$, then $f(0) > 1$. Since $\sigma(S) = J_c(f) \subset \overline{D(0,1)}$, then $f(0) \notin J_c(f)$ and therefore $0 \notin J_c(f)$. □

Proposition 4. *If* $p < 1/2$*, then* $E \bigcap J_c(f) \subset \partial J_c(f)$*.*
Proof:

Claim: $\forall n \in \mathbb{N}^*$, $\sigma(S_{2^n}) \bigcap J_c(f) = \emptyset$.

In effect, assume by absurd that there is an integer number $n \in \mathbb{N}$ and a complex number $z \in J_c(f)$ such that $f^n(z) = 0$. Since $0 \notin J_c(f)$, $f^{m+n}(z) = f^m(0)$ goes to infinity when m increase. But since $z \in J_c(f)$, $(f^{m+n}(z))_{m\geq 0}$ is bounded, we have a contradiction. Thus, the claim is true.

Hence,

$$\bigcup_{n=1}^{\infty} \sigma(S_{2^n}) \bigcap J_c(f) = \emptyset. \tag{7}$$

To prove the proposition we have to show that there is no element $z \in E \bigcap J_c(f)$ such that $z \in int(J_c(f))$. In fact, if there is such an element z, then there exists a subsequence (w_{n_i}) of 2 to 2 different elements of $\bigcup_{n=1}^{\infty} \sigma(S_{2^n})$ such that $\lim w_{n_i} = z$ and $f^{n_i}(w_{n_i}) = 0$.

Furthermore, since $z \in int(J_c(f))$, there is $r > 0$ such that $B(z, r) \subset J_c(f)$. Hence, since $\lim w_{n_i} = z$, there is a natural integer number N such that for all $i \geq N$, $w_{n_i} \in B(z, r) \subset J_c(f)$. This is a contradiction with (7). Then we are done. □

Demonstration of Theorem 3

i. From items ii. and iv. of proposition 0.3, we have $f^{-n}\{1\} \subset E$, for al $n \in \mathbb{N}$. Hence $\overline{\bigcup_{i=0}^{+\infty} f^{-n}\{1\}} \subset \overline{E} = E$, because the set of accumulation points is closed. Since $1 \in \partial J_c(f)$, we obtain by item iii. of Theorem 1.3.1 [12], $\partial J_c(f) = \overline{\bigcup_{n=0}^{\infty} f^{-n}\{1\}} \subset E$.

ii. Assume $p < 1/2$. Let prove that $E \subset \partial J_c(f)$.
Let $z \in E$ and suppose that $z \notin \partial J_c(f)$. Then, by proposition 0.4, we have $z \notin J_c(f)$. Therefore $\lim_{n\to+\infty} |f^n(z)| = +\infty$.

On the other hand, since $z \in E$, there exists a sequence of complex numbers $(w_{n_i})_{i\geq 0}$ converging to z and such that $f^{n_i}(w_{n_i}) = 0$, for all $i \in \mathbb{N}$.

Since $z \notin \partial J_c(f)$, then the sequence of functions $(f^n)_{n\geq 1}$ is normal in z. Then there is a subsequence $(n_i^{'})$ of (n_i) such that $f^{n_i^{'}}$ converges uniformly to a bounded analytic function g or $g = +\infty$ in some disk $B(z, r)$ with center $z \in \mathbb{C}$ and radius $r > 0$. That is,

$$\forall \varepsilon > 0, \exists\, N \in \mathbb{N},\ \forall\, i \geq N, \forall x \in B(z, r), |f^{n_i^{'}}(x) - g(x)| < \varepsilon.$$

In particular, if $x = w_{n_i^{'}}$, then $g(w_{n_i^{'}})$ converges to 0. But since $\lim w_{n_i} = z$ and g is a continuous function, then $g(z) = 0$. Thus g is bounded. But this can be truth because $\lim_{i\to\infty} |f^{n_i}(z)| = +\infty$. Hence $z \in \partial J_c(f)$ cause $|f^{n_i}(z)| \to \infty$.

iii. Assume that $p \geq 1/2$, then $0 \in J_c(f)$. Hence, for all integer $n \in \mathbb{N}$, $\sigma(S_{2^n}) \subset J_c(f)$.

Hence,

$$\bigcup_{n=1}^{\infty} \sigma(S_{2^n}) \subset J_c(f).$$

Therefore,

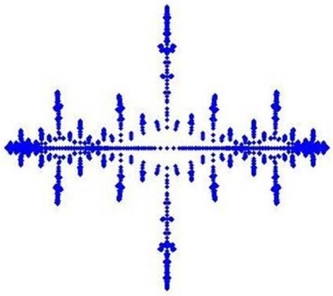
$$E = (\bigcup_{n=1}^{\infty} \sigma(S_{2^n}))' \subset (J_c(f))' = J_c(f).$$

□

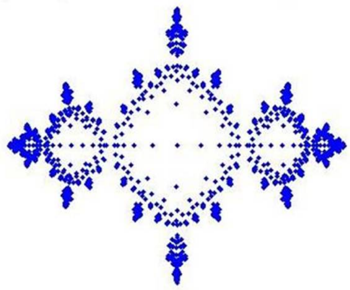
Observation 4. *For* $p = 1$ *we have* $E = \{0\}$ *and for* $p = 0$, $E = \{1\}$.

Conjecture. We conjecture that for $p \geq 1/2$, there exists $z \in J_c(f) \setminus \partial J_c(f)$ such that $z \in E$.

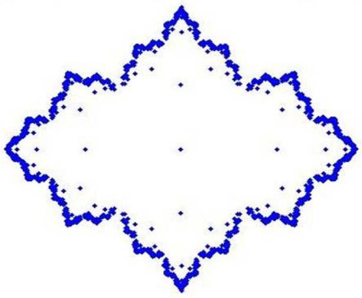
Pictures of E for $1/2 < p < 1$:



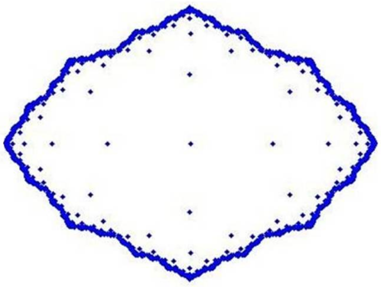
$p = 0.55$



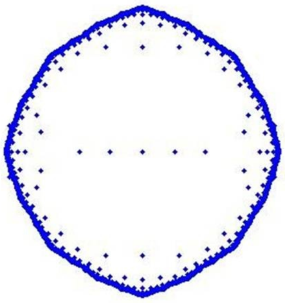
$p = 0.6$



$p = 0.7$



$p = 0.8$



$p = 0.9$

Pictures of E for $0 < p \leq 1/2$:

$p = 0.3$

$p = 0.38$

$p = 0.48$

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