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APROXIMACIÓN DE LA DISTANCIA EN EL PLANO A TRAVÉS DE LA SOLUCIÓN NUMÉRICA DE PROBLEMAS DE VALOR INICIAL ASOCIADOS A GEODÉSICAS

APPROXIMATION OF THE DISTANCE IN THE PLANE THROUGH THE NUMERICAL SOLUTION OF INITIAL VALUE PROBLEMS ASSOCIATED TO GEODESICS

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Resumen

En este artículo, se plantea un algoritmo para aproximar la distancia Euclidiana entre los puntos p y q del plano, mediante la búsqueda de geodésicas que salen del punto p y llegan a una vecindad del punto q. Esta búsqueda se hace a través de la solución numérica de problemas de valor inicial asociados a sistemas de ecuaciones diferenciales ordinarias de las geodésicas; para lo cual se elige un conjunto de direcciones en el plano.

Palabras Clave: Distancia Euclidiana, distancia intrínseca, distancia geodésica, geodésicas, problema de valor inicial, aproximación.

Abstract

In this paper, we proposes an algorithm to approximate the Euclidean distance between points $p\ and\ q$ of plane, by doing a search of geodesic that depart from the point p and arrive to a neighborhood of the point q. This search is done through the numerical solution of initial value problems associated with the system of ordinary differential equations of the geodesics; for this is choose a set of directions in the plane.

Keywords: Euclidean distance, intrinsec distance, geodesic distance, geodesics, initial value problem, approximation.

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1. Introduction

Determinate the distance between any two points is fundamental; because this value can be used in several applications in science. The formula for finding the Euclidean distance between points $p = (p_1, p_2)\ y\ q(q_1, q_2)$ is given by:

$$d(p,q) = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \|q - p\| \qquad (*)$$

where $q - p$ is the segment of straight line that joins the points $p\ and\ q$.

As the plane is a particular case of a Regular surface S; considering the points

$p, q \in S$, is obvious that the formula (*) not is valid when the curvature Gaussian is not null. In this case, the distance between the points p and q is given by the intrinsic distance or geodesic distance of S. In the plane the geodesics are straight lines; therefore the distance between points p and q, is the length of the geodesic segment (minimal) that connect these points.
In the general case of a complete Riemannian manifold, there are several methods to determine or approximating a minimal geodesic joining two points. Kaya and Noakes developed an Leap-Frog algorithm to approximate geodesic [1-2], Noakes developed a global algorithm [3]; and other methods are numerical approximations, using numerical methods to problems of contour [4], but in this case, there is no security if the solution approach to a minimal geodesic.
In this paper, we proposes an algorithm to approximate the Euclidean distance between points $p\ and\ q$ of plane, by doing a search of geodesic that depart from the point p and arrive to a neighborhood of the point q. This search is done through the numerical solution of initial value problems associated with the system of ordinary differential equations of the geodesics; for this is choose a set of directions in the plane.

In the plane, the canonical direction $v = q - p$, allow us to approximate the minimal geodesic that depart from the point p and arrive to point q; but in the case of an arbitrary regular surface, find this direction is not easy. Then, the search from a set of directions will allow to generalize the algorithm.

2. Set of Angles and Directions

In this section we will obtain results to formulate an algorithm to approximate Euclidean distances.
Consider the following sets

$$Ang = \{\alpha = \frac{n\pi}{180} \;/\; n = 1, \ldots, 360 \} \qquad (1)$$

$$Dir = \{ v = (cos\alpha, sen\alpha) \;/\; \alpha \in Ang \} \qquad (2)$$

Let $w_1, w_2 \in R^2$, $\|w_1\| = \|w_2\| = 1$, and $\theta = \measuredangle(w_1, w_2)$ the lower angle determined by w_1 y w_2, and $S_{w_1 w_2}$ the conical region determined by the vectors w_1, w_2, and let $\theta_v = \measuredangle vOX$ be the angle determined by the vector v and the positive axe X.

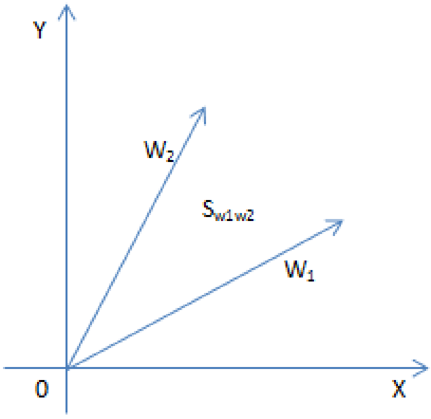

Fig. Nro1. Conical region $S_{w_1 w_2}$

Theorem 2.1. Let $\theta = 1°$, where $\theta = \measuredangle(w_1, w_2)$. We have:

a) If $w_1 \in Dir$, then $w_2 \in Dir$.

b) If $w_1 \notin Dir$, there is a $v \in Dir$ with $v \in int(S_{w_1 w_2})$, where int() denotes the interior of a set.

Proof

a) It is inmediate.

b) If $w_1 \notin Dir$, there are $v_1, v_2 \in Dir,\ v_2 = v_1 + 1$, with $w_1 \in int(S_{v_1 v_2})$. This implies that $v_2 \in int(S_{w_1 w_2})$.

□

Theorem 2.2. Let $\theta = 2°$, where $\theta = \measuredangle(w_1, w_2)$. There are two or three vectors in Dir that are in $S_{w_1 w_2}$.

Proof

a) If $w_1 \in Dir$, then $w_2 \in Dir$; and as $\theta = 2°$, there is $v \in Dir$, where v is bisectrix of angle θ. Therefore:

$$w_1, w_2, v \in S_{w_1 w_2}.$$

b) If $w_1 \notin Dir$, then $w_2 \notin Dir$. Therefore, there are $v_1, v_2 \in Dir,\ v_2 = v_1 + 1$, with $w_1 \in int(S_{v_1 v_2})$, this implies that $v_2 \in int(S_{w_1 w_2})$; and as $\theta = 2°$, there is $v \in Dir,\ v \neq v_2$, with $v \in int(S_{w_1 w_2})$.

Therefore:

$$v, v_2 \in int(S_{w_1 w_2}).$$

□

In general, if $\theta = n°$, there are n or n+1 vectors in Dir that belong to $S_{w_1 w_2}$, where $n \leq 359°$.

Definition 2.3. Let $q \in R^2,\ r \in R,\ r > 0$. The Circunference with center in **q** and radius **r**, is given by:

$$C(q,r) = \{\, x \in R^2 \,/\|x - q\| = r\}. \tag{3}$$

Definition 2.4. Let $q \in R^2,\ r \in R,\ r > 0$. The Circle with center in **q** and radius **r**, is given by:

$$Cir(q,r) = \{\, x \in R^2/\ \|x - q\| \leq r\}. \tag{4}$$

Let $p, q\ \in R^2, p \neq q$, and $d = d(p,q)$ the Euclidean distance between the points ***p*** and ***q,*** and consider the circle $Cir(q,r)$.

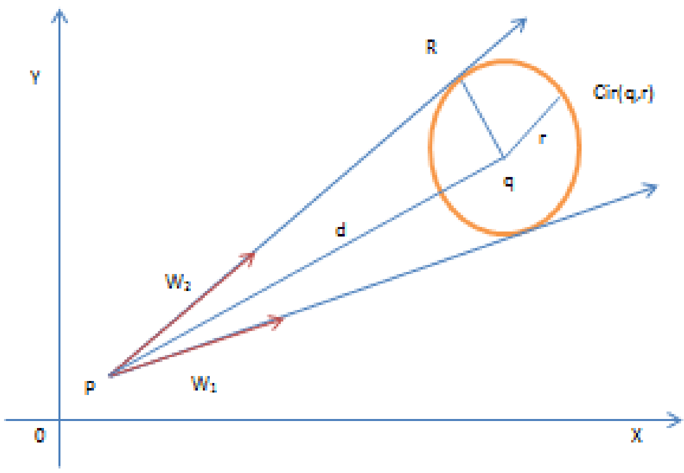

Fig. Nro. 2. Associates lines of $Cir(q,r)$.

Let L_1 *and* L_2 be the tangents lines to $C(q,r)$ drawn from point **p**. We consider the unit vectors generating w_1 *and* w_2 of the lines L_1 *and* L_2, respectively.
Without loss of generality, we can consider that $p = 0$, and q is on the positive semi axes X.

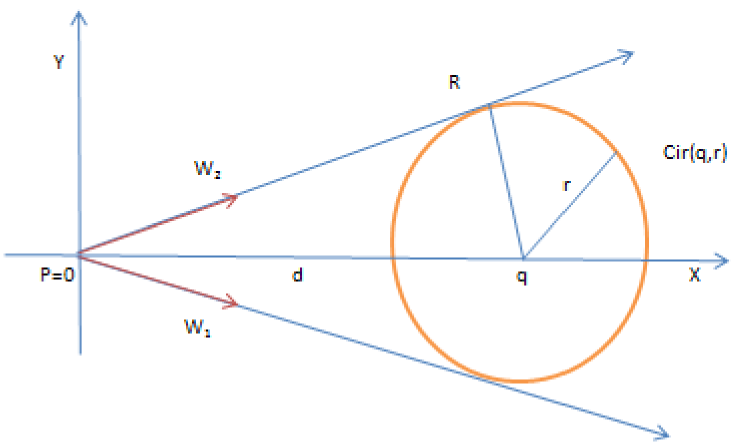

Fig. Nro. 3. Particular case on location of the points $p\ y\, q$.

Definition 2.5. Let $q \in R^2$, $r \in R$, $r > 0$. The circle $Cir(q, r)$ is called the Convergence Circle.

Theorem 2.6. Let $p, q \in R^2$, $p \neq q$, $p = 0$ and q be in the positive semi axes, $\theta = \measuredangle(w_1, w_2) = 1°$, $r = 0.1$. Then:

$$d(p, q) = d \approx 11.45930135.$$

Proof

Note that the triangle ΔqRO is a rectangle.

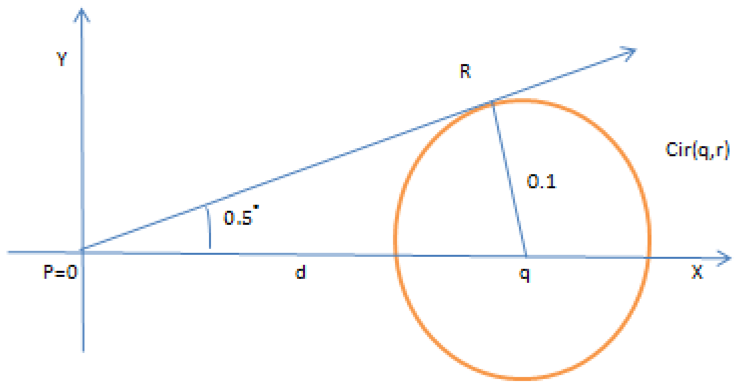

Fig. Nro. 4. Triangle determined by $Cir(q, r)$ and the point $p = 0$.

Since:

$$sen(0.5^{\circ}) = \frac{0.1}{d} = \frac{1}{10d}$$

$$d = \frac{1}{10sen(0.5^{\circ})}$$

Therefore:

$$d \approx 11.45930135.$$

□

Theorem 2.7. If $d = \frac{1}{10sen(0.5^{\circ})}$, $r = 0.1$, then there are one or two vectors in Dir, that are in $S_{w_1 w_2}$.

Proof

Since $d = \frac{1}{10sen(0.5^{\circ})}$, $r = 0.1$, then $\theta = \measuredangle(w_1, w_2) = 1°$. By theorem (2.1), there are one or two vectors in Dir that are in $S_{w_1 w_2}$.

□

Remark.

1. If $d = r = 0.1$, then $p = 0$ is the point of tangency, then there is a unique line tangent that is Y axes. Therefore, there are 180 directions in Dir that will be in $S_{w_1 w_2}$.
2. If $d < 0.1\ y\ r = 0.1$, then there are no points of tangency; therefore, they will have 360 directions in Dir that will be in $S_{w_1 w_2}$.

Definition 2.8. Let $r \in R,\ r = 0.1$. The Maximum distance and the Maximun Circle are given by:

$$d_{max} = \frac{1}{10sen(0.5^{\circ})} \tag{5}$$

$$C_R = \{x \in R^2 \ /\ \|x\| \leq R = \frac{d_{max}}{2}\}. \tag{6}$$

Theorem 2.9. For each pair of points $p, q \in C_R$, there is $v \in Dir$ which that $v \in S_{w_1 w_2}$, where $w_1\ y\ w_2$ are the generating of the tangents lines of $Cir(q, r)$ who go through p.

Proof

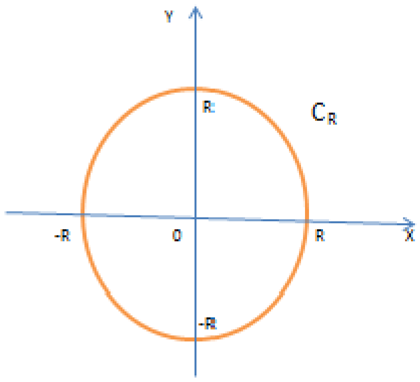

Fig. Nro. 5 Maximun Circle C_R centered at the origin.

a) Let $p, q \in C_R$, which that $d(p, q) = 2R = d_{max}$, by theorem (2.7), $\theta = \measuredangle(w_1, w_2) = 1^{\circ}$.

Therefore, exist one or two vectors $v \in Dir$ which are in $S_{w_1 w_2}$.

b) If $d(p, q) < 2R\ ,\ and\ p, q \in \partial C_R$, then $\theta = \measuredangle(w_1, w_2) > 1^{\circ}$, and there is $v \in Dir$, which are $v \in S_{w_1 w_2}$.

c) If $p, q \in int(C_R)$, then $d(p, q) < 2R$, and $\theta = \measuredangle(w_1, w_2) > 1^{\circ}$, therefore, there is $v \in Dir$, which that $v \in S_{w_1 w_2}$.

$\square$

3. Regular Surfaces.

In this section we enunciate some results on Differential Geometry, which were taken of the book of Do Carmo [5].

Definition 3.1. A subset $S \subset R^3$ is a Regular Surface if, for each $p \in S$, there exists a neighborhood V in R^3 and a map $X: U \subset R^2 \rightarrow V \cap S$ of an open set $U \subset R^2$ onto $V \cap S \subset R^3$ such that:

1. $X \in C^{\infty}(U)$.
2. X is a homeomorphism.
3. For each $q \in U$, the differential $dX_q: R^2 \rightarrow R^3$ is one-to-one.

The mapping X is called a parametrization of S; and in coordinates it is given for

$$X(u,v) = \big(x(u,v), y(u,v), z(u,v)\big), \forall (u,v) \in U.$$

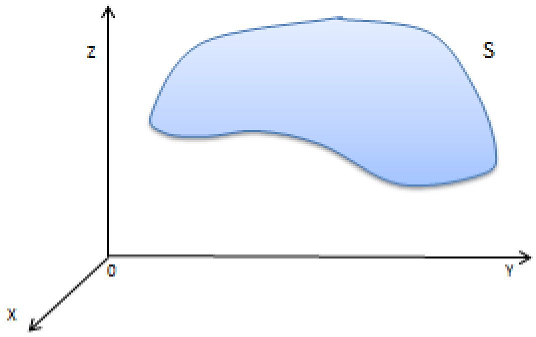

Fig. Nro. 6. The regular surface.

Definition 3.2. A nonconstant, parametrized curve $\alpha: I \subset R \rightarrow S$ is called parametrized Geodesic if:

$$\frac{D}{dt}\left(\frac{d\alpha}{dt}(t)\right) = 0, \forall\, t \in I, \qquad (7)$$

where $\frac{D}{dt}$ denotes the Covariant Derivative.

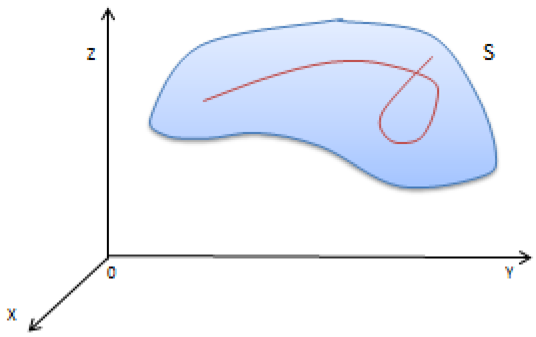

Fig. Nro. 7. The parametrized Geodesic on the regular surface.

Now, consider a parametrization $X: U \subset R^2 \to S$, which that $X(U) \cap \alpha(I) \neq \emptyset$. Also, the parametrization induce a base $\{X_u(q), X_v(q)\}$ in the tangent space T_pS, a S at $p = X(q)$.

Now, let $W(t)$ be a vectorial field tangent along a curve differentiable parametrized $\beta: I \subset R \to S$. The expressions of field $W(t)$ in the parametrization is:

$$W(t) = a(t)X_u\big(u(t), v(t)\big) + b(t)X_v\big(u(t), v(t)\big) \tag{8}$$

The expressions of covariant derivative of field $W(t)$, by (8), is:

$$\begin{aligned}\frac{D}{dt}W(t) = {}&(a' + \Gamma_{11}^1 au' + \Gamma_{12}^1 av' + \Gamma_{12}^1 bu' + \Gamma_{22}^1 bv')X_u \; + \\ &+ (b' + \Gamma_{11}^2 au' + \Gamma_{12}^2 av' + \Gamma_{12}^2 bu' + \Gamma_{22}^2 bv')X_v\end{aligned} \tag{9}$$

where the Γ_{ij}^k , $\forall\, i, j, k = 1{,}2$,are called the Christoffel symbols.

If $\alpha: I \subset R \to S$ is parametrized geodesic, its expressions in the parametrization is given by:

$$\alpha(t) = X(u(t), v(t)).$$

Therefore, the tangent vector is given by:

$$\frac{d\alpha}{dt}(t) = X_u u'(t) + X_v v'(t).$$

Using (7) and (9) for $W(t) = \frac{d\alpha}{dt}(t)$; we have:

$$\begin{cases} u'' + \Gamma_{11}^1(u')^2 + 2u'v'\Gamma_{12}^1 + \Gamma_{22}^1(v')^2 = 0 \\ v'' + \Gamma_{11}^2(u')^2 + 2u'v'\Gamma_{12}^2 + \Gamma_{22}^2(v')^2 = 0 \end{cases} \tag{10}$$

that is a system of ordinary differential equations of second order.

Let $p, q \in C_R$. Now, we approximate the Euclidean distance $d(p, q)$ using the system of differential equation of geodesic (10); for this we numerically solve initial value problems, using the Runge-Kutta method of fourth order.

Of (10), the I.V.P to solve numerically is:

$$\begin{cases} u'' + \Gamma_{11}^1(u')^2 + 2u'v'\Gamma_{12}^1 + \Gamma_{22}^1(v')^2 = 0 \\ v'' + \Gamma_{11}^2(u')^2 + 2u'v'\Gamma_{12}^2 + \Gamma_{22}^2(v')^2 = 0 \\ u(0) = u_0 \;\; , \;\; v(0) = v_0 \\ u'(0) = \tau_1 \;\; , \;\; v'(0) = \tau_2 \end{cases} \tag{11}$$

where $\tau = (\tau_1, \tau_2) \in R^2$, $\|\tau\| = 1$.

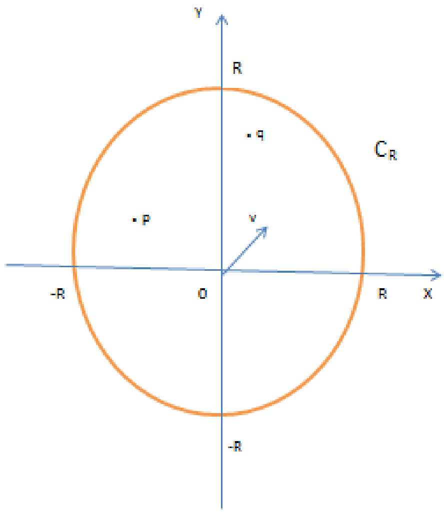

Fig. Nro. 8. Initial value problem on C_R.

4. Algorithm.

In this section we developed a algorithm to approximate the Euclidean distance $d(p,q)$, $p,q \in C_R$, doing a search through the numerical solution of I.V.P (11).

1. Input $p = (p_1, p_2),\ q = (q_1, q_2),\ and\ N.$

2. $h = \frac{1}{N}$, $t_i = ih$, $i = 0,1,\dots,N.$

   $u(0) = p_1$, $v(0) = p_2$, $tol = 0.1.$

3. For i of 1 to 360, do

   $$\theta_i = \frac{i\pi}{180}$$

4. For i of 1 to 360, do

   $\tau_1 = cos\theta_i$, $\tau_2 = sin\theta_i$

   $u'(0) = \tau_1$, $v'(0) = \tau_2$

   Solver the I.V.P (11),

   $Error = \sqrt{(u(k) - q_1)^2 + (v(k) - q_2)^2}$

   4.1. If $Error \leq tol$, then

   $l_i \leftarrow$ length of the i- geodesic.

   Save l_i .

5. $d_a \leftarrow mín\{l_i\}$, where d_a is the distance obtained by the algorithm.

   $d(p,q) \approx d_a.$

7. End

Remark.

If $p, q \in C_R$, $r = 0.1$, by theorem (2.9), there exist $v \in Dir$ such that $v \in S_{w_1 w_2}$.

5. Examples.

5.1. Estimate the Euclidean distance between two points $p, q \in C_R$, $r = 0.1$, using the Algorithm given in the previous section.

p		q		Distance: d_a	Euclidean Distance: d	Error $\lvert d - d_a \rvert$
1.0	1.0	2.0	3.0	2.25	2.236068	0.0139
-4.0	2.0	3.0	3.0	7.05	7.07106	0.0210
-4.8	-2.7	3.0	-4.0	7.95	7.9076	0.0424
3.0	-4.0	-4.8	-2.7	7.95	7.9076	0.0424
-2.0	-5.0	1.0	5.3	10.65	10.7280	0.0780
-2.0	-5.0	-4.8	-2.7	3.60	3.6235	0.0235
5.4	-1.0	0.0	-4.0	6.15	6.1774	0.0726
-3.8	-2.7	4.3	1.2	9.00	8.9900	0.0100
2.5	-3.7	-1.3	2.1	6.90	6.9340	0.0340
4.8	3.9	-3.2	-2.9	10.50	10.4995	0.0005

Table Nro. 1. Comparison between the distance d_a and distance Euclidean d.

5.2. Estimate the Euclidean distance between two points $p, q \forall \in C_R$, $r = 0.1$, where C_R is centered in the point $(7, 6)$, using the algorithm given in the previous section.

p		q		Distance: d_a	Euclidean Distance: d	Error $\|d - d_a\|$
4.0	2.0	9.0	10.0	9.45	9.43400	0.015
3.1	1.9	3.0	9.8	7.95	7.9006	0.0494
2.8	7.9	6.5	6.9	3.90	3.8328	0.0672
11.7	3.2	3.5	7.9	9.45	9.4515	0.0015
7.2	5.6	9.4	7.8	3.15	3.1113	0.0387
4.7	4.2	7.9	10.2	6.75	6.8000	0.0500
5.3	5.1	8.3	6.4	3.30	3.2696	0.0304
4.4	4.3	3.2	6.8	2.70	2.7731	0.0731
8.3	6.4	9.2	10.2	3.90	3.9051	0.0051
5.9	4.7	2.9	3.7	3.15	3.1623	0.0123

Table Nro. 2. Comparison between the distance d_a and distance Euclidean d.

6. Conclusions

If the distance between two points is less or equal to d_{max}, then always there will be $v \in Dir$ for approximate the minimal geodesic; another case, it could never happen an vector in Dir for approximate the minimal geodesic. Therefore, the sets Ang and Dir must be refined.

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